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Research Article

FRACTIONAL DIFFERENTIAL EQUATIONS AND FRACTIONAL SEMIGROUPS OF OPERATORS

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ABSTRACT

In this paper, the authors investigate the solution of a second-order fractional differential equation. They consider cases where either the variable u or v is missing. The paper also introduces the concept of a fractional semigroup of operators. This involves a mapping that represents bounded linear operators on a Banach space U. The one-parameter semigroup satisfies certain properties, such as the semigroup property and having an identity operator at y = 0. The derivative of this semigroup at y = 0 is called the infinitesimal generator. The main objective of the paper is to explore the fundamental properties of these fractional semigroups and their connection to the fractional derivative of the semigroup at y = 0. We confirm that we are using the definition of Al-Sharif and Malkawi [22] for the conformable fractional derivative.

Keywords: conformable fractional derivative, fractional β−semigroup, c0 − semigroup, β − infinitesimal generator.

INTRODUCTION

Fractional differential equations are a fascinating area of study in mathematics. They involve derivatives of non-integer order, which adds a new level of complexity to traditional differential equations. These equations are used to model various phenomena in science and engineering, such as viscoelastic materials, electrical circuits, and biological systems. Fractional differential equations provide a more accurate representation of real-world processes that exhibit memory and long-range dependence.

Solving fractional differential equations requires specialized techniques, such as fractional calculus and fractional operators. Researchers have developed different numerical and analytical methods to tackle these equations and understand their behavior. The study of fractional differential equations has applications in many fields, including physics, biology, finance, and control systems. It offers a deeper understanding of complex systems and contributes to the development of innovative solutions.

Fractional calculus has become a captivating field in mathematical analysis. The concept originated from a question posed by L'Hospital in 1695 [15]. Researchers have since attempted to generalize the traditional derivative to the fractional derivative. Numerous definitions have been proposed, many of which utilize integral forms [9, 10, 13, 17, 18, 19]. However, there have been inconsistencies among existing fractional derivatives. For instance, not all fractional derivatives satisfy the familiar product and quotient rules for derivatives, and most of them, except for the Caputo derivative, do not yield a derivative of zero for constant functions.

You can see many important results made by several authors [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 16, 20, 21]. In 2014, Khalil and others introduced an intriguing definition of the fractional derivative known as the conformable fractional derivative, which employs a limit approach.

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Definition 1.1. [11] *Let f : [0, ∞) → R be a function. The βth order "conformable fractional derivative" of f is defined by*

$$T_\beta(f)(y) = \lim_{\epsilon\to 0}\frac{f(y+\epsilon y^{1-\beta})-f(y)}{\epsilon}$$

for all y > 0 and β ∈ (0, 1). If f is β−differentiable in some (0, a), a > 0, and $\lim_{y\to 0^+} f^\beta(y)$ *exists, then we define* $f^\beta(0) = \lim_{y\to 0^+} f^\beta(y)$.

Following Khalil, Katugampola gave a new definition that generalizes Definition 1 as follows:

Definition 1.2. [8] Let f : [0, ∞) → R and y > 0. *Then the fractional derivative of order β is defined by*

$$D_\beta(f)(y) = \lim_{\epsilon\to 0}\frac{f(te^{\epsilon y^{-\beta}})-f(y)}{\epsilon}$$

for all y > 0 and β ∈ (0, 1). If f is β−differentiable in some (0, a), a > 0, and $\lim_{y\to 0^+} f^\beta(y)$ *exists, then we define* $f^\beta(0) = \lim_{y\to 0^+} f^\beta(y)$.

Definition 2 is a more general version of Definition 1 because it includes truncating the series $ye^{\epsilon y^{-\beta}} = \sum_{k=0}^{\infty}\frac{\epsilon^k y^{1-\beta k}}{k!}$ when k = 1, we obtain the formula in Definition 1. In 2020, Al-Sharif and Malkawi [22] defined a new modification of the conformable fractional derivative with classical properties.

Definition 1.3. [22] Let f : [0, ∞) → R and y > 0. *Then the fractional derivative of f of order β is defined by,*

$$M^{(\beta)}(f)(y) = \lim_{\epsilon\to 0}\frac{f(yg(\epsilon y^{-\beta}))-f(y)}{\epsilon},$$

where g is a continuously differentiable function such that g(0) = g'(0) = 1.

If g(y) is taken to be g(y) = 1 + y, we get Definition 1 and if $g(y) = e^y$, we get Definition 2. The purpose of this section is to present some properties of the gamma function; a very important function in mathematics and statistics. The gamma function is a generalization of the factorial function and defined as follows: for β > 0,

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx. \quad (1.1)$$

The function Γ(β) is convex for positive real numbers and

$$\Gamma(\beta) = (\beta - 1)\Gamma(\beta - 1). \quad (1.2)$$

Using (1.2), the gamma function for β = n ∈ N, turns to be the factorial function:

$$\Gamma(n) = (n - 1)!. \quad (1.3)$$

Theorem 1.1. [22] Let a ∈ (0, 1] and f, L be β−differentiable at a point y > 0. Then,

(1) $M^{(\beta)}[af + bL] = aM^{(\beta)}(f) + bM^{(\beta)}(L)$, *for all* $a, b \in R$.

(2) $M^{(\beta)}(y^n) = ny^{n-\beta}$, *for all* $n \in R$.

(3) $M^{(\beta)}(C) = 0$, *for all constant functions,* $f(y) = C$.

(4) $M^{(\beta)}(fL) = LM^{(\beta)}(f) + fM^{(\beta)}(L)$.

(5) $M^{(\beta)}(\frac{f}{L}) = \frac{LM^{(\beta)}(f) - fM^{(\beta)}(L)}{L^2}$.

(6) If in addition, f is differentiable, then $M^{(\beta)}(f)(y) = y^{1-\beta}\frac{df}{dy}$.

Theorem 1.2. [22] Let a, n ∈ R and β ∈ (0, 1]. Then, we have the following results.

(a) $M^{(\beta)}(e^{au}) = au^{1-\beta}e^{au}$.

(b) $M^{(\beta)}(\sin(au)) = au^{1-\beta}\cos(au)$.

(c) $M^{(\beta)}(\cos(au)) = -au^{1-\beta}\sin(au)$.

(d) $M^{(\beta)}(\frac{1}{\beta}t^{\beta}) = 1$.

We also have rather unusual results given in the next theorem.

Theorem 1.3. [22] . Let β ∈ (0, 1] and y ∈ R. Then,

(i) $M^{(\beta)}(\sin(\frac{1}{\beta}y^{\beta})) = \cos(\frac{1}{\beta}y^{\beta})$.

(ii) $M^{(\beta)}(\cos(\frac{1}{\beta}y^{\beta})) = -\sin(\frac{1}{\beta}y^{\beta})$.

(iii) $M^{(\beta)}(e^{\frac{1}{\beta}y^{\beta}}) = e^{\frac{1}{\beta}y^{\beta}}$.

Definition 1.4. (Fractional Integral). Let a ≥ 0 and y ≥ 0. Also, Let f be a function defined on (a, y] and β ∈ R . Then, the β − Fractional Integral of f is defined by

$$I_a^{\beta}(f)(y) = \int_a^y \frac{f(x)}{x^{1-\beta}} dx.$$

If the Riemann improper integral exists.

Theorem 1.4. [22] (Inverse property). Let a ≥ 0, y ≥ 0 and let f be a continuous function such that $I^{\beta}{}_{a}$ f exists. Then

$$M^{(\beta)}(I_a^{\beta} f)(y) = f(y), \text{ for } y \geq a.$$

Theorem 1.5. [22] Let β ∈ (0, 1] and φ, ψ be β − F ractional Integral on (a, y], 0 ≤ a < y. Then,

(1) $I_a^{\beta}(b\varphi + c\psi)(y) = bI_a^{\beta}(\varphi)(y) + cI_a^{\beta}(\psi)(y)$, *for all* $a, b \in R$.

(2) $I_a^{\beta}(y^n) = \frac{1}{(n+\beta)}[y^{n+\beta} - a^{n+\beta}]$, *for all* $n \in R$.

(3) $I_a^{\beta}(C) = \frac{C}{\beta}[y^{\beta} - a^{\beta}]$, *for all constant functions,* $\varphi(y) = C$.

In this paper, we investigate the solution of a second-order fractional differential equation of the form F (u; v; $v^{(\beta)}$; $v^{(2\beta)}$) = 0, in the cases where either u is missing or v is missing.

Let U be a Banach space, and Q : [0, ∞) ⟶ L(U, U) be a mapping that represents bounded linear operators on U. A family {Q(y)}y>0 ⊆ L(U, U) is referred to as a one-parameter semigroup if it satisfies Q(s + y) = Q(s)Q(y), and Q(0) = I, where I is the identity operator on U. The derivative of the semigroup at y=0 is known as the infinitesimal generator of the semigroup.

Remark 1.1. *Whenever we mention the notation "M^{β}" in this paper, it refers to the conformable fractional derivative defined by Al-Sharif and Malkawi [22].*

FRACTIONAL DIFFERENTIAL EQUATIONS

In the context of solving a second-order fractional differential equation, let's discuss the cases where either the variable u or v is missing. When solving a second-order fractional differential equation, it is not uncommon to encounter situations where one of the variables, either u or v, is missing. This means that the equation only involves derivatives of one variable, while the other variable is absent.

In such cases, the solution strategy may differ depending on which variable is missing. If u is missing, the equation can be simplified to a fractional ordinary differential equation involving only the variable v. Similarly, if v is missing, the equation can be reduced to a fractional ordinary differential equation involving only the variable u.

The solution of these simplified equations can be approached using various techniques, such as Laplace transforms, power series methods, or fractional calculus approaches. The specific method chosen depends on the nature of the equation and the desired form of the solution.

It is important to note that the absence of one variable in a second-order fractional differential equation does not necessarily imply a loss of generality or significance. In fact, such cases can provide valuable insights into the behavior and properties of fractional differential equations. I hope this sheds some light on handling situations where either u or v is missing in the solution of a second-order fractional differential equation.

Let's think about a fractional differential equation of the form:

$$F(u, v, v^{(\beta)}, v^{(2\beta)}) = 0, \quad (2.1)$$

In this case, we have $v^{(\beta)}$ as the β-conformable derivative of v with respect to u, where β ∈ (0, 1]. Additionally, $v^{(2\beta)} = M^{\beta} M^{\beta} v$. It's worth noting that this equation is not a standard one that we can easily handle.

The object of this paper is to try to solve equation (1) in case either u is missing or v is missing using what we will call fractional reduction of order. There are two cases to be considered

(i) $F(u, v, v^{(\beta)}, v^{(2\beta)}) = 0$, v is missing

(ii) $F(u, v, v^{(\beta)}, v^{(2\beta)}) = 0$, u is missing

Case (i) v is missing

In this case, let's set $v^{(\beta)}$ as w. As a result, we have $v^{(2\beta)} = w^{(\beta)}$. This reduces the equation's order from 2β to β, which is much more manageable.

Example 2.1. $v^{(2\beta)} - (v^{(\beta)})^2 = 1$

You're right, this equation is not linear. However, in this case, v is missing. So, let's set $v^{(\beta)}$ as w and as a result, we have $v^{(2\beta)} = w^{(\beta)}$. This transforms the equation into...

$$w^{(\beta)} = w^2 + 1,$$

we can solve this equation as a separable differential equation:

Since w(β) = $u^{1-\beta}$ dw/du, [3], the equation $w^{(\beta)}$ = w2 + 1 can be written as:

$$u^{1-\beta}\frac{dw}{du} = 1 + w^2.$$

Thus tan−1(w) = 1/βx^{β} + β. Consequently, w = tan(1/β x^{β} + β). Replacing w by $v^{(\beta)}$ and then substituting $v^{(\beta)}$ by u(1−β) du/dv and integrating we get:

$$v = -\ln\left|\cos(\frac{1}{\beta}u^{\beta} + a)\right| + b, \text{ a,b are constants.}$$

Example 2.2. $4u^{\beta-1}(\cos u)v^{(2\beta)} - (\sin u)(v^{(\beta)})^2 = 4 \sin u$.

since v is missing, let's substitute $v^{(\beta)}$ with w. As a result, we have $v^{(2\beta)}$ = w(β). This transforms the equation into

$$4u^{\beta-1}(\cos u)w^{(2\beta)} - (\sin u)w^2 - 4\sin u = 0.$$

The equation we have now is a separable differential equation:

$$\frac{1}{w^2+4}dw = \frac{\sin u}{\cos u}du,$$

which can be solved to get

$$w = 2\tan 2(c_1 - \ln|\cos u|).$$

Replacing w by $v^{(\beta)}$ and then substituting $v^{(\beta)}$ by $u^{(1-\beta)}$ dx/dy, we get:

$$v = 2u^{\beta-1}\tan 2(c_1 - \ln|\cos u|)du$$

Case (ii)

In this case we put $v^{(\beta)}$ = w. Then $v^{(2\beta)}$ = $M_u^{\beta}w = u^{1-\beta} M_{vw}Muv = v^{(\beta)}M_v w = wM^v w$, we use $M^{\beta}{}_u w$ to represent the fractional derivative of w with respect to u. Similarly, we can do the same for v.

So, by making this substitution, we can simplify the equation and express it as a new equation with lower order in terms of w and v.

Example 2.3. Consider $vv^{(2\beta)} + (v^{\beta})^2 = 0$.

$$Put\ v^{(\beta)} = w.\ So\ v^{(2\beta)} = v^{(\beta)}\tfrac{dw}{dv}.$$

Hence

$$vv^{(\beta)}\frac{dw}{dv} + w^2 = vw\frac{dw}{dv} + w^2 = 0.$$

Solving this equation to get:

vw = n. Thus vv(β) = b,

This equation can be rewritten as vdv = buβ−1du, which is another separable equation. We can solve this equation to find the solution

$$v^2 = \frac{2b}{\beta}u^{\beta} + c, \quad b, c\ are\ constant.$$

FRACTIONAL SEMIGROUPS OF OPERATORS

In this section, the concept of a fractional semigroup of operators is introduced. This concept involves a mapping that represents bounded linear operators on a Banach space U. The key feature of this one - parameter semigroup is that it satisfies certain properties, such as the semigroup property and having an identity operator at y=0.

One of the important aspects of studying fractional semigroups is understanding their derivative at y=0, which is referred to as the infinitesimal generator. The infinitesimal generator provides valuable insights into the behavior and properties of the semigroup.

The main objective of the section is to delve into the fundamental properties of these fractional semigroups and establish their connection to the fractional derivative of the semigroup at y=0.7 By exploring these properties, researchers aim to enhance our understanding of the dynamics and characteristics of fractional semigroups.

Definition 3.1. *For a Banach space U, a family of operators {Q(y)}y≥0 ⊆ (U, U) is referred to as a fractional β-semigroup (or β-semigroup) if β ∈ (0, ∞] and a > 0 for all a:*

$$(i)\ Q(0) = I,$$
$$(ii)\ Q(s+y)^{\frac{1}{\beta}} = Q(s^{\frac{1}{\beta}})Q(y^{\frac{1}{\beta}}), \forall s, y \in [0,\infty).$$

Clearly, if β = 1, then 1-semigroup are just the usual semigroups.

Example 3.1. We can define the space X as C[0, ∞), which represents the set of real-valued continuous functions on the interval [0, ∞).

Now, let's define (Q(y)φ)(s) = φ(s + 2√y). By showing that Q satisfies the properties of a 1/2- semigroup of operators, we can easily demonstrate that it is indeed a 1/2- semigroup.

Definition 3.2. An β − semigroup Q(y) is called a a c0 − semigroup if, for each fixed x ∈ X, Q(y)u ⟶ u as y⟶ 0^+.

The congormable β − derivative of Q(y) at y = 0 is called the β − infinitesimal generator of the fractional β − semigroup Q(y), with the domain equals

$$\left\{u \in U : \lim_{y\to 0^+} M^{\beta}(Q(y))u\ \ exists\right\}.$$

We will write A for such generator.

Theorem 3.1. Let {Q(y)}y≥0 ⊆ (U, U) be a c0 − β − semigroup with infinitesimal generator A, 0 < β < 1. If Q(y) is continuously β − differentiable and u ∈ Dom(A), then

$$M^{\beta}(Q(y)u = AQ(y)u = Q(y)Ax.$$

Proof. Let's get started with

$$\begin{aligned} M^{\beta}(Q(y))u &= \lim_{\varepsilon\to 0}\frac{Q(yg(\varepsilon y^{-\beta})))u - Q(y)u}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{Q(y^{\beta} + (yg(\varepsilon y^{-\beta})))^{\beta} - y^{\beta})^{\frac{1}{\beta}}u - Q(y)u}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{Q(y^{\beta} + ((yg(\varepsilon y^{-\beta})))^{\beta} - y^{\beta}))^{\frac{1}{\beta}}u - Q(y)u}{\varepsilon}. \end{aligned}$$

If $Q(y)$ is indeed a β-semigroup of operators, then $Q(a+b)^{\frac{1}{\beta}} = Q(a^{\frac{1}{\beta}})Q(b^{\frac{1}{\beta}})$. Hence

$$\begin{aligned} M^{\beta}(Q(y))u &= \lim_{\varepsilon\to 0}\frac{Q(y^{\beta})^{\frac{1}{\beta}}Q((yg(\varepsilon y^{-\beta})))^{\beta} - y^{\beta}))^{\frac{1}{\beta}}u - Q(y)u}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{Q(y)[Q((yg(\varepsilon y^{-\beta})))^{\beta} - y^{\beta}))^{\frac{1}{\beta}}u - Q(0)u}{\varepsilon}. \end{aligned}$$

Now, using the Mean Value Theorem for conformable fractional derivative, see [6], we get

$$\frac{Q(y)[Q((yg(\varepsilon y^{-\beta}))^{\beta}-y^{\beta}))^{\frac{1}{\beta}}u-Q(0)u}{\varepsilon}=Q(y)M^{\beta}(Q(c))u\frac{(yg(\varepsilon y^{-\beta}))^{\beta}-y^{\beta}}{\beta\varepsilon}$$

for some $0 < c < (yg(\varepsilon y^{-\beta}))^{\beta} - y^{\beta}$.

If $\varepsilon \to 0$, then $c \to 0$, and $\lim_{\varepsilon\to 0} M^{\beta}(Q(c)) = M^{\beta}(Q(0)) = A$. Consequently,

$$M^{\beta}(Q(y))u = Q(y)Au \lim_{\varepsilon\to 0}\frac{(yg(\varepsilon y^{-\beta}))^{\beta}-y^{\beta}}{\beta\varepsilon}.$$

Using L'Hopital's Rule, we get

$$\lim_{\varepsilon\to 0}\frac{(yg(\varepsilon y^{-\beta}))^{\beta}-y^{\beta}}{\beta\varepsilon}=1.$$

Hence

$$M^{\beta}(y)u = Q(y)Au.$$

Similarly one can show that $Q(y)u \in Dom(A)$ and $M^{\beta}(Q(y))u = AQ(y)u$.

Theorem 3.2. The in nitesimal generator of the above semigroup is

$$\begin{aligned} A\varphi(s) &= \varphi'(s), \\ Dom(A) &= \{\varphi \in U : \varphi' \ exists\ in\ U\}. \end{aligned}$$

Proof.

$$\begin{aligned} M^{\beta}(Q(y)\varphi(s) &= y^{1-\beta}Q'(y)\varphi(s) \\ &= y^{1-\beta}\lim_{\varepsilon\to 0}\frac{Q(y+\varepsilon+O(\varepsilon^2))\varphi(s)-Q(y)\varphi(s)}{\varepsilon} \\ &= y^{1-\beta}\lim_{\varepsilon\to 0}\frac{\varphi(s+\frac{1}{\beta}(y+\varepsilon)^{\beta})-\varphi(s+\frac{1}{\beta}y^{\beta})}{\varepsilon} \\ &= y^{1-\beta}\lim_{\varepsilon\to 0}\frac{\varphi(s+\frac{1}{\beta}(y+\varepsilon)^{\beta})-\varphi(s+\frac{1}{\beta}y^{\beta})}{\varepsilon}\cdot\frac{\varphi(s+y+\varepsilon)-\varphi(s+y)}{\varphi(s+y+\varepsilon)-\varphi(s+y)} \\ &= y^{1-\beta}\lim_{\varepsilon\to 0}\frac{\varphi(s+\frac{1}{\beta}(y+\varepsilon)^{\beta})-\varphi(s+\frac{1}{\beta}y^{\beta})}{\varphi(s+y+\varepsilon)-\varphi(s+y)}\cdot\frac{\varphi(s+y+\varepsilon)-\varphi(s+y)}{\varepsilon}. \end{aligned}$$

Now,

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{\varphi(s+y+\varepsilon)-\varphi(s+y)}{\varepsilon} &= \varphi'(s+y), \\ \lim_{\varepsilon\to 0}\varphi(s+\frac{1}{\beta}(y+\varepsilon)^{\beta})-\varphi(s+\frac{1}{\beta}y^{\beta})\varphi(s+y+\varepsilon)-\varphi(s+y) &= \frac{0}{0}. \end{aligned}$$

Using L'Hopital rule (with respect to ε) to get

$$\lim_{\varepsilon\to 0}\frac{\varphi(s+\frac{1}{\beta}(y+\varepsilon)^{\beta})-\varphi(s+\frac{1}{\beta}y^{\beta})}{\varphi(s+y+\varepsilon)-\varphi(s+y)}=\frac{y^{\beta-1}\varphi'(s+\frac{1}{\beta}y^{\beta})}{\varphi'(s+y)}.$$

Thus the product gives

$$M^{\beta}(Q(y))\varphi(s)=\varphi'(s+\frac{1}{\beta}y^{\beta}).$$

Hence $A\varphi = \varphi'$.

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